Specialist Investigation

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1 Introduction

Mathematics is the study of the unknown, the way in which mankind systematically questions dark uncertainty and unifies knowledge under one universal language. It is not about memorising formulas or passing exams but answering questions yet unsolved, that is the purpose of this assessment and the driving force behind my response, curiousity.

This investigation details three puzzles and proposed solutions: Russian Cubes, Bell Ringing and Inclusion/Exclusion principle. These problems relate primarily to basic combinatorics and counting techniques but also require a level of critical thinking and problem solving capabilities, all these puzzles were given with no solutions or aid material so the following method has not yet been verified nor revised.

2 Russian Cubes

2.1 Introduction

The Russian cubes problem begins with a simple question, a cube is painted with 3 blue faces and three red faces, how many different cubes can be painted like this? Really this question is asking for the total possible permutations of faces on the cube, however the question also implies that all reds are identical and all blues are identical, knowing a cube has 6 faces we can propose this equation for the total possible different permutations:

$$\frac{6!}{3! \cdot 3!} = 20 \tag{1}$$

Therefore the total permutations of colours on the cube is 20, this can be shown by drawing cube nets and colouring them in.

2.2 Correcting Initial Assumptions

The answer from 2.1 fails to fully encompass the puzzle, the question implies that once painted the cube can be rotated. This rotation, not represented in flat nets nullifies many of our previously thought 'different permutations' as the cube can be rotated to form an equivalent pattern even if it is created from a different net. Create your own nets and cubes and you will quickly realise there are less than 20 solutions. After preforming this task many times and creating two Lego cube models it was realised that a cube can be separated into 3 pairs of opposite sides. These individual sides can be re-arranged within their respective pair and the pairs can be re-arranged in order from top to bottom resulting in the number of permutations for the pairs:

$$2! \cdot 2! \cdot 2! \cdot 2! \cdot 3! = 48 \tag{2}$$

At first this seems to imply a greater number of permutations, which we know should be lower than 20 because of the cube's free rotation, this is due to the forgotten statement that all reds are identical and all blues are identical represented by the expression:

$$3! \cdot 3! \tag{3}$$

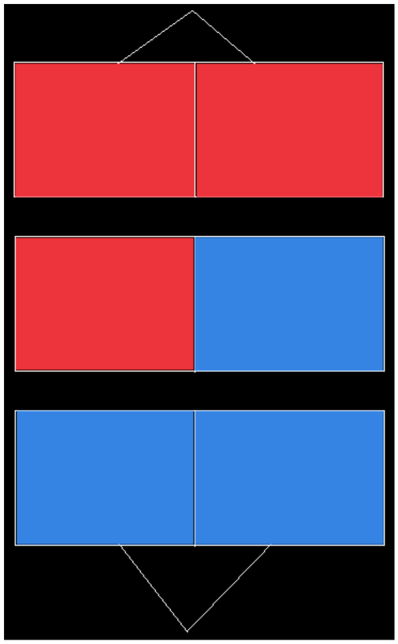
Putting the new number of permutations over this expression, to nullify all repeated cases where reds are swapped with other reds and blues are swapped with other blues:

$$\frac{2! \cdot 2! \cdot 2! \cdot 2! \cdot 3!}{3! \cdot 3!} = \frac{4}{3} \tag{4}$$

2.3 Final Solution

This too is clearly not the answer, the number of permutations cannot be anything but a positive integer. This issue arises because the denominator is too large, meaning we are trying to eliminate too many reoccurring permutations. Upon further thought and model creation a flaw was discovered with this equation. In cases where 2 reds and 2 blues are in the same pair (refer to image below, arrows indicating identical colours already accounted for) this equation defines the identical nature of all reds and all elements twice with the same two elements. Therefore the denominator must be $3! \cdot 3! -$ all cases where 2 reds and 2 blues are in the same pair or $2!^3 \cap 3! \cdot 3!$. The intersection is therefore $2! \cdot 3!$, as the pairs can be arranged in 3! different orders and the only pair without identical colours can be re-arranged 2! ways (RB and BR). Therefore the final proposed solution is:

$$\frac{2!^3 \cdot 3!}{3! \cdot 3! - 2! \cdot 3!} = 2 \tag{5}$$



2.4 Conclusion

The answer of 2 possible permutations reached in 2.2 seems reasonable, logically if the position of individual elements does not matter nor their order then there can only be two different nets. This finding answers the second problem posed in the Russian Cubes Section:

James (using blue) and Cathy (using red) paint the faces of a cube in turn so that the six faces are painted in order 'blue then red then blue then red then blue then red',

Having finished one cube they begin to paint the next one.

Show that, even though she always goes second, Cath can choose the faces she paints in a way that will ensure that both cubes are identical.

Quite simply as long as Cath paints the opposite faces of the cube the same set of colours as the previous one she will be able to create identical cubes. She can do this by always painting the opposite face to the one that James has just painted. If the first cube has a set of 2 opposite red faces and 2 opposite blue faces Cath must paint her first two opposite sides the same colour as James'. If the first cube has no opposite pairs of the same colour she must always paint her face a different colour to James'.

3 Bell Ringing

3.1 Introduction

The second problem in this investigation is "Bell Ringing", best detailed in the brief given:

Suppose you are a bell-ringer holding a rope and you look around the church tower and see the faces of 3 friends, all about to start change ringing. To ring a 'round' each bell is rung in turn (1234,1234,1234...). The bells can be rung in any order and changing the order is known as a change. As your bell goes round on its wheel you can slow it down, or speed it up, just a little but not much, so you can only change places in the ringing order with the bell just before you or just after you.

By these rules NRICH can at first change to RNICH but not to RINCH. What are the other orders of the letters of the word NRICH that can be obtained in just one change of this sort?

The following example shows very simple 'bell music' starting with a round and ending with a round of 4 bells, showing 8 of the 24 possible permutations, or orders. 1234 2143 2413 4231 4321 3412 3142 1324 1234

Can you find the changes so that, starting and ending with a round, all the 24 possible permutations are rung once each and only once?

3.2 Method

This question at first seems arduous, running through all 24 possible combinations with no repetition and finishing in the same order that was started from. The first point to realise is that the starting combination does not matter, any 4 number sequence is equal but for the purpose of simplicity the order '1234' was chosen. The reasoning behind the problem is exceedingly complex and very obscure at the beginning, so to begin all 24 possible permutations were written out in no particular order to help give some clarity to the final objective:

$$\begin{gathered}
1234\\
1243\\
1423\\
1432\\
1324\\
1342\\
2134\\
2143\\
2314\\
2341\\
2413\\
2431\\
3124\\
3142\\
3214\\
3241\\
3412\\
3421\\
4123\\
4132\\
4213\\
4231\\
4312\\
4321
\end{gathered} \tag{6}$$

After writing out these permutations it was realised each number (1-4) occupies every position 6 times i.e 1 will be in the first position 6 times, 1 will be in the second position six times... This means that it would be possible to cycle through all 24 permutations only once in an order starting with 1234 by cycling through all the permutations with 1 in the first position and then 3 and then 4, finishing with 2. Logically 2 must be the last cycle, working backwards from the final set 1234 we can only move each digit over by one position and so the second last set could be 2134 or 2143 since we are starting by cycling through all the sets with 1 in the first position. Although these regulations are not imposed by the question they let us start from somewhere and even if the first solution does not work, we will likely know how to construct a working second attempt.

3.3 Proposed Solution

After working within these parameters the following order was devised:

$$
\begin{gathered}
1234\\
1243\\
1423\\
1432\\
1342\\
1324\\
-\\
3142\\
3124\\
3214\\
3241\\
3421\\
3412\\
-\\
4321\\
4312\\
4132\\
4123\\
4213\\
4231\\
-\\
2413\\
2431\\
2341\\
2314\\
2134\\
2143\\
-\\
1234
\end{gathered}
\tag{7}
$$

This order of bell sets fulfils the given requirements, cycling through all permutations, finishing on the first set and not repeating any set twice.

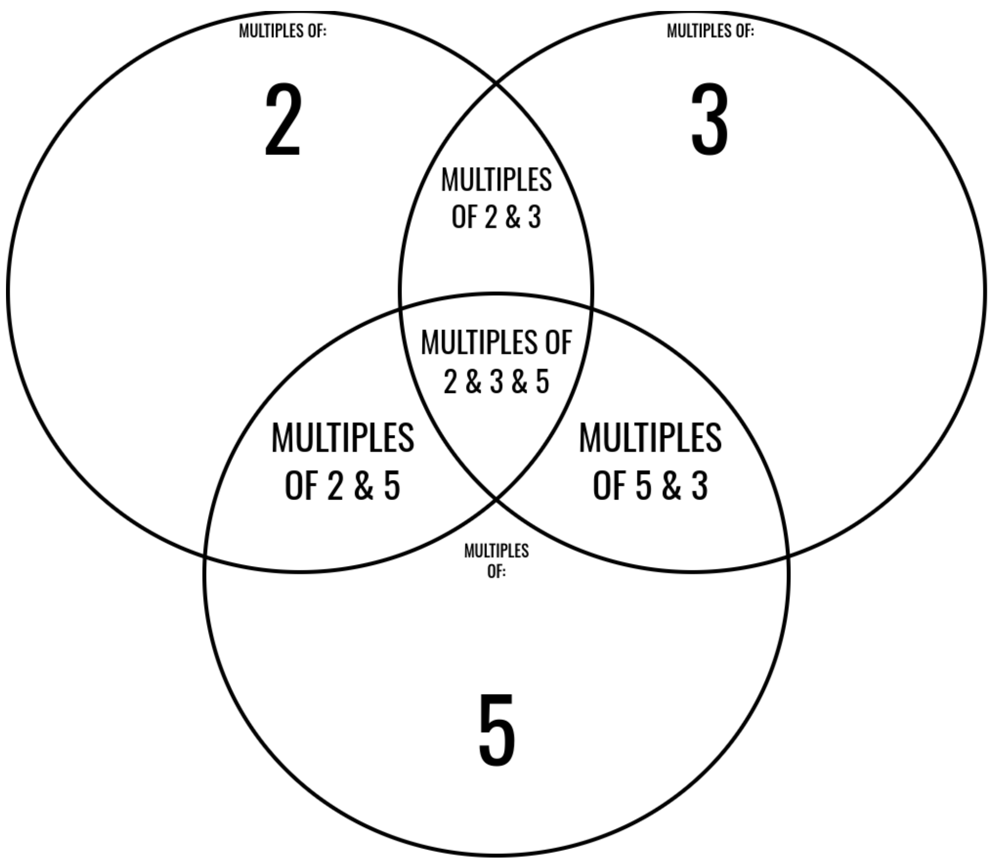
3.4 Conclusion

Although the final solution was reached without much difficulty by breaking the problem into pieces and setting extra parameters the underlying logic of the order was never uncovered. However there is an interesting pattern in the changing of bell ringing orders: within each 6 set cycle only 2 digits were rotated each round. I.e in the first round '34' became '43' but '12' remained the same. It was only in the crossover between cycles that both pairs needed to be rotated to avoid repetition, i.e '1324' became '3142'. In this case without moving two pairs in the next cycle 2 would have to be in the first position and not 4, as was required.

4 Inclusion/exclusion

4.1 Introduction

This problem poses the simple question:*How many integers between 1 and 1200 are not multiples of any of the numbers 2,3 or 5?* Though the question seems simple the answer is rather obscure, and as an aid the hint was given to utilise a Venn-diagram. This alludes to the point that there is an overlap between multiples of 2,3 and 5. This is visualised in the diagram below:


MULTIPLES OF:
2
MULTIPLES OF:
3
MULTIPLES OF 2 & 3
MULTIPLES OF 2 & 3 & 5
MULTIPLES OF 2 & 5
MULTIPLES OF 5 & 3
MULTIPLES OF:
5

4.2 Method

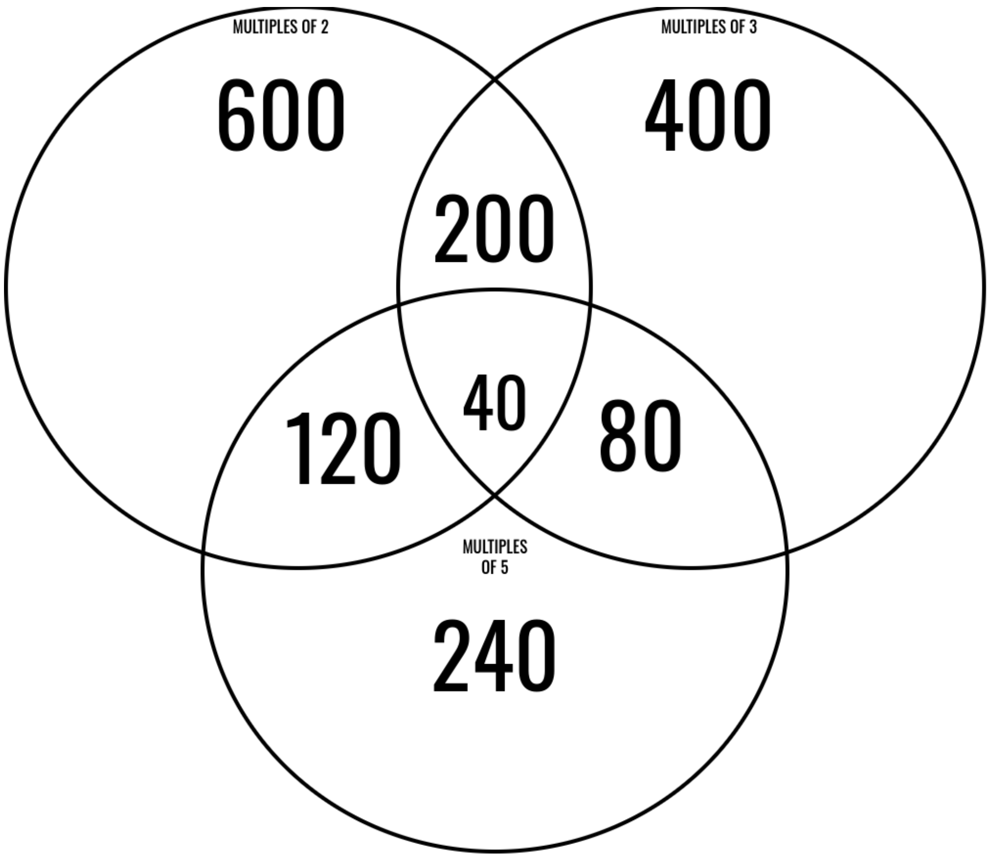
By subtracting 1200 (total integers) by the total area of the Venn-diagram shape we will determine the number of integers that lie outside the Venn-diagram and therefore the number of integers that are not multiples of 2,3 or 5. First we must determine the area of each individual circle by determine how many integers it contains:

$$\begin{aligned} For2. \frac{1200}{2} &= 600 \\ For3. \frac{1200}{3} &= 400 \\ For5. \frac{1200}{5} &= 240 \end{aligned} \tag{8}$$

Now the area of all the intersections must be calculated:

$$\begin{aligned} For2 \cap 3. \frac{1200}{2 \cdot 3} &= 200 \\ For2 \cap 5. \frac{1200}{2 \cdot 5} &= 120 \\ For3 \cap 5. \frac{1200}{3 \cdot 5} &= 80 \\ For2 \cap 3 \cap 5. \frac{1200}{2 \cdot 3 \cdot 5} &= 40 \end{aligned} \tag{9}$$

Notice we calculate the intersections by using the LCM of the two/three integers, in this case it is just the product of the integers as they are all prime numbers. Now view the updated Venn-diagram below:


MULTIPLES OF 2
MULTIPLES OF 3
600
400
200
40
120
80
MULTIPLES
OF 5
240

4.3 Final Solution

To calculate the area of this Venn-diagram we must take the sum of all the circle areas and then take away the area of each intersection once excluding the intersection of all three, then remove this intersection of all 3 multiples twice. Think about it as if we are removing all the excess layers that the circles form on one another until the entire diagram is one layer thick. If this explanation is still confusing create 3 overlapping circles for yourself out of paper and work out how you would remove each portion systematically to find the total area of the shape.

$$\begin{gathered} A = 600_{multiples\ of\ 2} + 400_{multiples\ of\ 3} + 240_{multiples\ of\ 5} \\ -(200-40)_{2\cap 3\ without\ 2\cap 3\cap 5} \\ -(120-40)_{2\cap 5\ without\ 2\cap 3\cap 5} \\ -(80-40)_{3\cap 5\ without\ 2\cap 3\cap 5} \\ -(40\cdot 2)_{2\cap 3\cap 5} \\ A = 880 \end{gathered} \tag{10}$$

If this is the area of the Venn-diagram then by subtracting 1200 by this area we should determine the total integers between 1 and 1200 that are not multiples of 2,3 or 5: $1200 - 880 = 320$

5 Conclusion of Findings

To conclude, the solutions reached for these problems are not definitive, they are my own workings and have not been reviewed by my peers or compared with the known answers. I have only outlined my logic in what I hope is clear and simple to read LaTeXdocument. My workings were severely limited by my unfamiliarity with the nuances of combinatorics and my naivety in proper mathematical presentation, my solutions rely too heavily on explanations and general logic and don't clearly outline the mathematics. Though there is no simple solution for improving this I believe that with every assignment and assessment I become slightly more articulate in the art that is maths.

6 Reflections and Techniques Learnt

During this investigation the solutions and methodology were discussed for each problem given but in reality it involved a great deal more. This was my first attempt at using the LaTeXsoftware to structure an assignment. I have spent a great deal of time learning the format and ASCII symbols to structure the equations and sections of this paper, learning slowly how to utilise an important professional document. By slowly working through these puzzles and detailing my working I discovered that the art of methodically approaching and reviewing problems without outside aid or others answers. Both the Russian Cubes and Inclusion/Exclusion dilemmas taught me to visualise problems and create practical models, whether that be out of paper circles or Lego cubes. The Bell Ringing question taught me to break difficult problems into smaller steps and first clearly define what the solution encompasses, even if that means writing out every permutation until I see a pattern.